MATH 3235 Probability Theory

11/1/22

Cauchy-Schwarz Inequality

$$\mathbb{E}(X) = \mathbb{E}(Y) = 0$$

$$\operatorname{cov}(X, Y) = \mathbb{E}(XY)$$

$$\rho_{X,Y} = \frac{\operatorname{cov}(X, Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X = \sqrt{\mathbb{E}(X^2)}$$

$$\rho_{aX, bY} = \rho_{X,Y}$$

Th:

$$-1 \le \rho_{X,Y} \le 1$$

if $\rho_{X,Y} = 1 \implies Y = aX + b$, $a > 0$

$\rho_{X,Y} = -1 \implies Y = aX + b$, $a < 0$

Th:

$$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\,\mathbb{E}(Y^2)$$

$$|\mathbb{E}(X,Y)| \leq \sqrt{\mathbb{E}(X^2)\,\mathbb{E}(Y^2)}$$

Proof:

$$\mathbb{E}\left((X - a Y)^2\right) \geq 0 \quad \forall a$$

$$\parallel$$

$$h(a) = \mathbb{E}(X^2) - 2a\,\mathbb{E}(XY) + a^2\,\mathbb{E}(Y^2)$$

$$h'(a) = 2a\,\mathbb{E}(Y^2) - 2\,\mathbb{E}(XY)$$

$$h'(a) = 0 \implies \bar{a} = \frac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$$

$$h(\bar{a}) = \mathbb{E}(X^2) - 2\,\frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)} + \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)}$$

$$= \mathbb{E}(X^2) - \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)} \geq 0$$

q. e. d.

$\vec{x} \quad \vec{y} \in \mathbb{R}^n$

$|(\vec{x}, \vec{y})| \leq \|\vec{x}\| \, \|\vec{y}\|$

$\mathbb{E}(XY)$ is a kind of scalar product between X and Y.

$(\vec{x}, \vec{y}) = \|\vec{x}\| \, \|\vec{y}\|$ if and only if $\vec{x} = \lambda \vec{y}$ for some λ.

$\mathbb{E}(XY)^2 = \mathbb{E}(X^2)\,\mathbb{E}(Y)^2$ iff $X = aY$.

$$\rho_{XY} = \frac{\mathbb{E}((X-\mu_X)(Y-\mu_Y))}{\sqrt{\mathbb{E}((X-\mu_X)^2)\,\mathbb{E}((Y-\mu_Y)^2)}}$$

$\mu_X = \mathbb{E}(X) \qquad \mu_Y = \mathbb{E}(Y)$

Suppose X with $\mathbb{P}(X \geq 0) = 1$.

t a given level

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$$

Proof:

$$A = \{ \omega \mid X(\omega) \geq t \}$$

$\omega \in A \qquad X(\omega) \geq t$

$\omega \notin A \qquad X(\omega) \geq 0$

$$X \geq t \, \mathbb{1}_A$$

$$\mathbb{1}_A(\omega) = \begin{cases} 0 & \omega \notin A \\ 1 & \omega \in A \end{cases}$$

$$\mathbb{E}(X) \geq \mathbb{E}(t \, \mathbb{1}_A) \geq t \, \mathbb{E}(\mathbb{1}_A) =$$

$$= t \, \mathbb{P}(A) = t \, \mathbb{P}(X \geq t)$$

Chebyshev's Inequality

$$\mathbb{P}(|X - \mu_X| \geq t) \leq \frac{1}{t^2} Var(X)$$

$$(X-\mu_x)^2 = Z$$

$$\mathbb{P}(Z \geq t^2) \leq \frac{1}{t^2} \mathbb{E}(Z) \qquad \text{Markov}$$

$$\mathbb{P}((X-\mu_x)^2 \geq t^2) \leq \frac{1}{t^2} \operatorname{Var}(X)$$

X_i i.i.d. $\qquad \mathbb{E}(X_i) = \mu$

$\qquad \operatorname{var}(X_i) = \sigma^2$

$$\frac{1}{N} \sum_i X_i = \bar{X}$$

$$\mathbb{E}(\bar{X}) = \frac{1}{N} \sum_i \mathbb{E}(X_i) = \mu$$

$$\operatorname{var}(\bar{X}) = \frac{1}{N^2} \sum_i \operatorname{var}(X_i) = \frac{\sigma^2}{N}$$

$$\mathbb{P}(|X_i - \mu| \geq t) \leq \frac{\sigma^2}{t^2}$$

$$\mathbb{P}(|\bar{X} - \mu| \geq t) \leq \frac{\sigma^2}{N t^2}$$

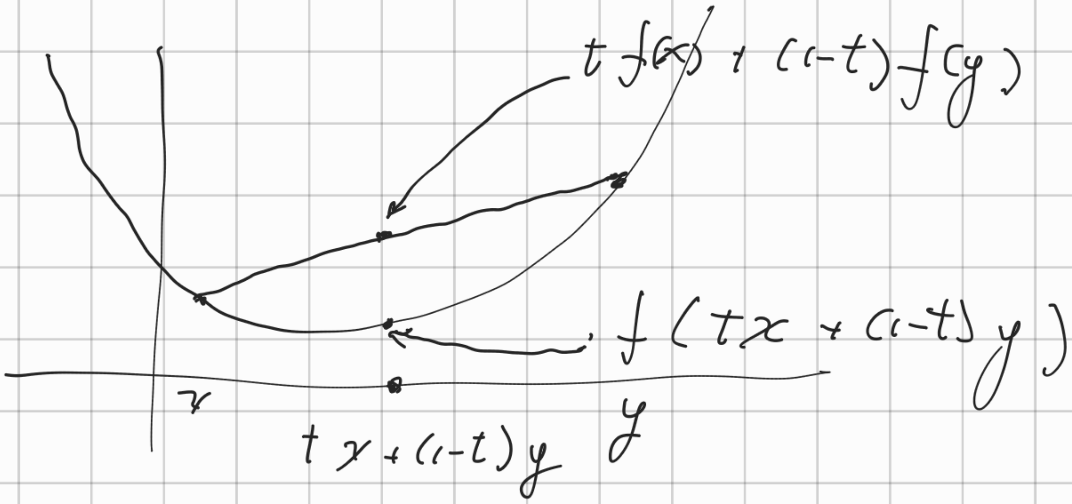
Jensen inequality

Convex function

$f(x)$ is convex if

$\forall$ x, y and $0 \le t \le 1$

$$f(tx + (1-t)y) \le t f(x) + (1-t) f(y)$$



$t \in [0, 1]$ $\qquad x \le tx + (1-t)y \le y$

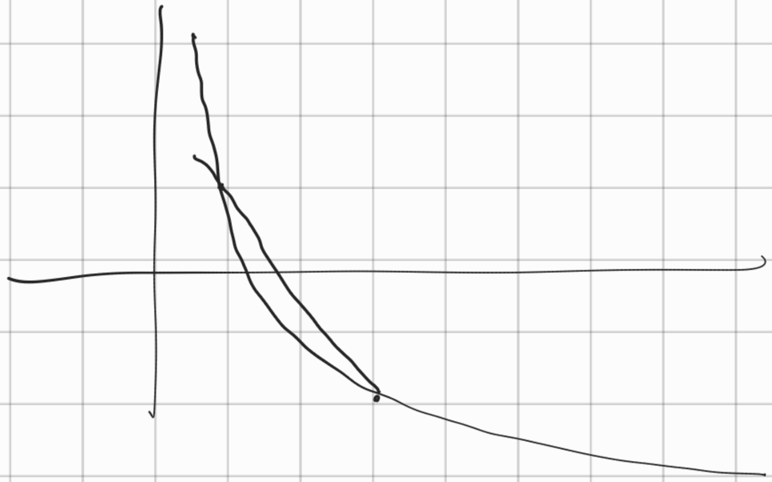
f is convex if $f''(x) > 0$

if

x, y and $t \in (0, 1)$

$$f(tx + (1-t)y) < t f(x) + (1-t) f(y)$$

strictly convex.



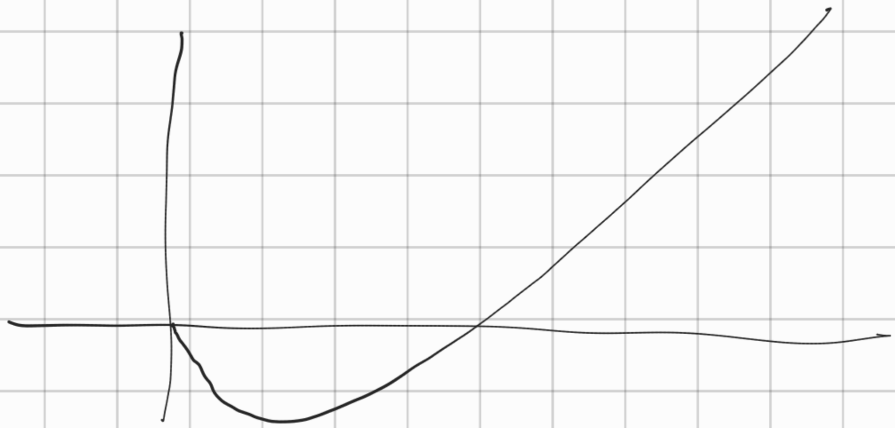
$-\log x$

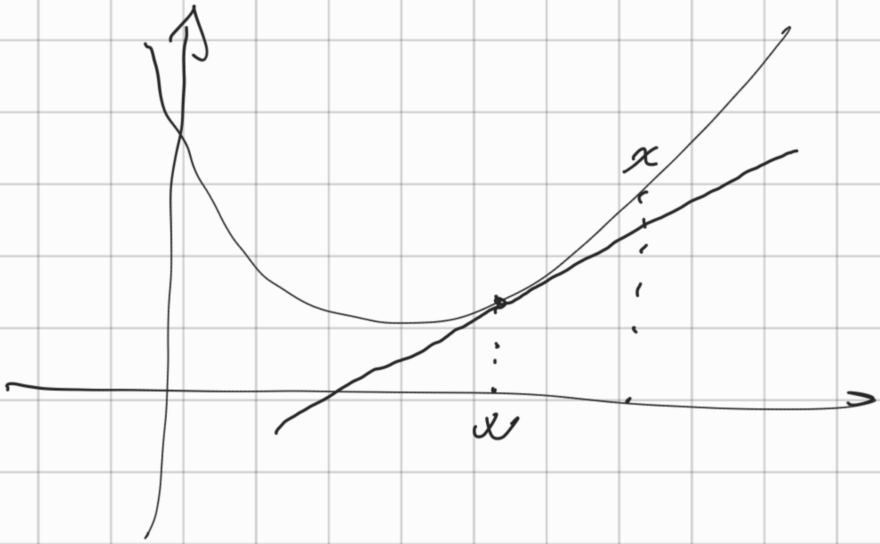
$$\frac{d}{dx^2}\left(-\log x\right) = \frac{1}{x^2}$$

$f(x) = x \log x$ is convex

$$f'(x) = 1 + \log x$$

$$f''(x) = \frac{1}{x}$$





w

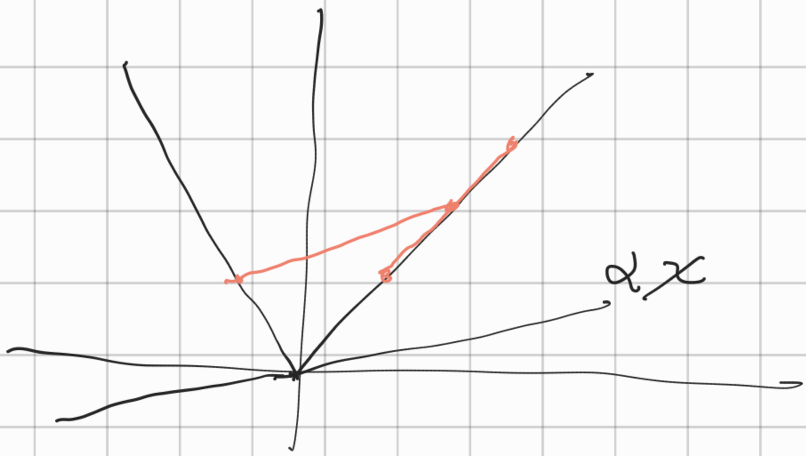
$\forall x \quad g(x) \geq g(w) + g'(w)(x-w)$

Th. if g is convex

$\forall w \quad \exists \alpha$ such that

$$g(x) \geq g(w) + \alpha(x-w)$$

$\forall x$.



$|x|$

$|\alpha| < 1$

Jensen inequality

if g is convex

$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$

Proof: $\mu = \mathbb{E}(X)$

$$g(x) \geq g(\mu) + \alpha(x-\mu)$$

$$g(X) \geq g(\mu) + \alpha(X-\mu)$$

$$\mathbb{E}(g(X)) \geq g(\mu) + \alpha\cancel{(\mathbb{E}(X)-\mu)}$$

$$\mathbb{E}(g(X)) \geq g(\mathbb{E}(X))$$

X is a r.v. that takes value x with prob t and value y with prob $(1-t)$

Then Jensen ineq. $\Rightarrow$ definition of convexity.

$$\mathbb{E}(\log X) \leq \log \mathbb{E}(X)$$

X is a discrete r.v. with values x_i $\quad i = 1 \dots n$

$$\mathbb{P}(X = x_i) = 1/n$$

$$\mathbb{E}(\log X) = \frac{1}{n} \sum_i \log x_i$$

$$\mathbb{E}(X) = \frac{1}{n} \sum_i x_i$$

$$\log \left(\prod_i x_i \right)^{1/n} \leq \log \frac{1}{n} \sum_i x_i$$

$$\left(\prod_i x_i \right)^{1/n} \leq \overline{x}$$